Now substituting for θ and w from Eqs. (21) and (23) in Eq. (19), multiplying the resultant expression by $\sin n\pi z$, and integrating over z, we obtain a system of linear homogeneous equations for the constants $C_m/(m^2\pi^2+k^2)$; and the requirement that these constants not all be zero leads to the secular equation in simplified form

$$\Bigg\| -\frac{\delta_{mn}}{2Rk^2D_n^3}+2\delta mn\pi(1-(-1)^{m+n})D_nD_m+\Big\{-2kn\pi D_n^2(1+(-1)^{n+1}\cosh k)+\frac{\delta}{2}\Big[2n\pi D_n^2(-1)^{n+1}\sinh k+4kn\pi D_n^2 \times(-1)^{n+1}\cosh k-8k^2n\pi^2D_n^3(-1)^{n+1}\sinh k\Big]-\frac{\delta}{k}(-1)^{n+1}n\pi D_n^2 k\sinh k\Big\}A_2^{(m)}+\Big\{-2kn\pi D_n^2(-1)^{n+1}\sinh 2k +\frac{\delta}{2}\Big(4kn\pi D_n^2(-1)^{n+1}\sinh k+2n\pi D_n^2[1+(-1)^{n+1}\cosh k]-8k^2n\pi D_n^3[1+(-1)^{n+1}\cosh k]\Big) -\delta n\pi D_n^2[1+(-1)^{n+1}\cosh k]\Big\}B_2^{(m)}+\frac{1}{2}\delta_{mn}-\delta X_{nm}\Bigg\|=0 \tag{29}$$

where

$$X_{nm}=0$$

if $m+n$ is even and $m\neq n$,

$$X_{nm}=\frac{1}{4} \tag{30}$$

if $m=n$,

$$X_{nm}=\frac{4mn}{(n^2-m^2)}\left[\frac{1}{m^2\pi^2+k^2}-\frac{1}{\pi^2(n^2-m^2)}\right]$$

if $m+n$ is odd, and

$$D_j=(j^2\pi^2+k^2)^{-1} \tag{31}$$

A first approximation to the solution of Eq. (29) is obtained by setting the (1,1) element of the matrix equal to zero. Thus, we find after making use of the expressions for $A_2^{(1)}$, $B_2^{(1)}$

$$R=(1+\delta/2)(\pi^2+k^2)^3/k^2\left(1-\frac{16k\pi^2\cosh^2 k/2}{(\sinh k+k)(\pi^2+k^2)^2}\right) \tag{32}$$

We observe that, apart from the factor $(1+\delta/2)$, this expression for R is identical to what was obtained by Pellew and Southwell[2] for the simple Bénard problem by the variational method in the first approximation for the case of two rigid boundaries. Consequently,

$$R_c=(1+\delta/2)\times 1715, \qquad k_{\min}=3.12$$

Here, also, we observe that if δ is positive, i.e., the viscosity is increasing upwards, the critical Rayleigh number is increased, and if δ is negative R_c is decreased, similar to the case of free boundaries when $\mu=1+\delta z$.

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Computation of Flow Past a Rotating Cylinder with an Energy-Dissipation Model of Turbulence

B. I. Sharma*

Imperial College of Science and Technology, London, England

Nomenclature

$C_\mu, C_1, C_2, C_3, C_c$	= constants in turbulence model
eff	= effective value
k	= turbulent kinetic energy
ℓ	= length scale of turbulent eddy
p	= static pressure
r	= radial distance from the axis of symmetry
R	= radius of cylinder
R_t	= turbulent Reynolds number, $k^2/\nu\epsilon$
Re_∞	= freestream Reynolds number, $U_\infty R/\nu$
Ri	= swirl Richardson number, defined by Eqs. (2) and (11)
U	= velocity in the x direction
U_∞	= freestream velocity
V_θ	= circumferential velocity
W	= velocity directed normal to the surface
x	= coordinate measured along the surface
z	= coordinate measured normal to the surface

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*Dept. of Mechanical Engineering; presently Staff Engineer at Union Carbide Corporation, Tonawanda, N. Y.

δ	= boundary-layer thickness where $U/U_\infty = 0.99$
$\delta_{2\theta}$	$= \int_0^\delta U/U_\infty [1-(U/U_\infty)] dz$, momentum thickness in the x direction
δ_{2x}	$= \int_0^\delta UV_\theta / U_\infty \omega R dz$, swirl momentum thickness
ϵ	= dissipation rate of turbulence energy
$\sigma_{k,\epsilon}$	= Prandtl numbers for diffusion of k and ϵ
ρ	= density of fluid
μ	= dynamic viscosity
ν	= kinematic viscosity
ω	= angular velocity of cylinder
Ω	= swirl velocity/freestream velocity

Introduction

FINITE difference calculations of flow past a rotating cylinder[1] were recently compared with the only available experimental data of Furuya et al.[2] and Parr.[3] The model of turbulence employed in Ref. 1 was a version of the Prandtl's mixing length hypotesis (MLH). The effect of streamline curvature there was taken into account by making a length scale of turbulence (mixing length) that prevails in plane flows a linear function of the local "swirling flow" Richardson number, i.e.,

$$\ell = \ell_0 (1 - \beta Ri) \tag{1}$$

where ℓ_0 is the level of mixing length that would prevail in plane flows in the absence of swirl. Ri is the swirl flow Richardson number

$$Ri = (2V_\theta / r^2)(\partial/\partial z)(rV_\theta) / (\partial U/\partial z)^2 + (r_{\dot{y}} \partial V_\theta / \partial z)^2 \tag{2}$$

and β with a value of five was found to give the best overall agreement of predictions with experiment. In this Note, the direct effect of streamline curvature is incorporated into the energy-dissipation model of turbulence by an empirical coefficient whose magnitude is directly proportional to the ratio of the time scale for significant rotational distortion to that of the energy-containing turbulent motions.

Mathematical and Physical Model

The governing equations in the axial and circumferential directions for a uniform property, axisymmetric turbulent flow may be written

$$\rho U \frac{\partial U}{\partial x} + \rho W \frac{\partial U}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{r}\frac{\partial}{\partial z}\left[r\mu_{\text{eff}} \frac{\partial U}{\partial z}\right] \tag{3}$$

$$\rho U \frac{\partial (rV_\theta)}{\partial x} + \rho W \frac{\partial (rV_\theta)}{\partial z} = \frac{1}{r}\frac{\partial}{\partial z}\left[r^3 \mu_{\text{eff}} \frac{\partial (V_\theta / r)}{\partial z}\right] \tag{4}$$

which, together with the equation of continuity

$$\frac{\partial (rU)}{\partial x} + \frac{\partial (rW)}{\partial z} = 0 \tag{5}$$

and of radial equilibrium of the mean motion

$$(\partial p / \partial z) = (\rho V_\theta^2 / r) \tag{6}$$

constitute a closed set. The independent variables x and z are, respectively, the axial and normal directions. The corresponding velocities are U and W. V_θ denotes the circumferential velocity. All symbols are defined in the nomenclature. The effective viscosity of the fluid μ_{eff} is taken as the sum of the laminar and turbulent contributions, i.e.

$$\mu_{\text{eff}} = \mu + \mu_t \tag{7}$$

In the present work, the turbulent viscosity μ_t is obtained from the solution of the following differential and auxiliary equations

$$\mu_t = C_\mu \rho k^2 / \epsilon \tag{8}$$

Turbulence kinetic energy k and its local rate of dissipation ϵ are, respectively, given by

$$\rho U \frac{\partial k}{\partial x} + \rho W \frac{\partial k}{\partial z} = \frac{1}{r}\frac{\partial}{\partial z}\left[r\left(\frac{\mu_t}{\sigma_k} + \mu\right)\frac{\partial k}{\partial z}\right] + \mu_t\left[\left(\frac{\partial U}{\partial z}\right)^2 + \left(r\frac{\partial V_\theta / r}{\partial z}\right)^2\right] - \rho\epsilon - 2\mu\left(\frac{\partial k^{1/2}}{\partial z}\right)^2 \tag{9}$$

$$\rho U \frac{\partial \epsilon}{\partial x} + \rho W \frac{\partial \epsilon}{\partial z} = \frac{1}{r}\frac{\partial}{\partial z}\left[r\left(\frac{\mu_t}{\sigma_\epsilon} + \mu\right)\frac{\partial \epsilon}{\partial z}\right] + C_1 \frac{\epsilon \mu_t}{k}\left[\left(\frac{\partial U}{\partial z}\right)^2 + \left(r\frac{\partial V_\theta / r}{\partial z}\right)^2\right] - C_2 \frac{\rho \epsilon^2}{k}(1 - C_c Ri) + C_3 \nu \mu_t \left\{\frac{\partial}{\partial z}\left[\left(\frac{\partial U}{\partial z}\right)^2 + \left(r\frac{\partial V_\theta / r}{\partial z}\right)^2\right]^{1/2}\right\}^2 \tag{10}$$

where

$$C_\mu = 0.09 \exp[-3.4/(1 + R_t/50)^2]$$

$$C_2 = 1.92[1.0 - 0.3\exp(-R_t^2)]$$

and $R_t = \rho k^2 / \mu\epsilon$ is the turbulent Reynolds number. The other empirical coefficients take the following values

$$C_1 = 1.44 \quad C_c = 0.2 \quad \sigma_k = 1.0 \quad \sigma_\epsilon = 1.3$$

Boundary conditions are applied at the cylinder surface ($z = 0$) and beyond the edge of the boundary layer ($Z = Z_\infty$) as follows

$$z = 0 \quad U = k = \epsilon = 0 \quad V_\theta = \omega r$$

$$Z = \alpha \quad U = U_\infty \quad k = \epsilon = V_\theta = 0$$

The above equations for turbulence energy k and its rate of dissipation ϵ differ from that in Ref. 4. Extra source terms involving gradients of (V_θ / r) appear in Eqs. (9) and (10). Their appearance is due to the conversion of the Cartesian tensor form of these equations to the present coordinate system.

The transport equation for the rate of dissipation (10) contains a new source term embodying the effect of streamline curvature. Ri is again the Richardson number

$$Ri = \frac{k^2}{\epsilon^2}\frac{V_\theta}{r^2}\frac{\partial (rV_\theta)}{\partial r} \tag{11}$$

wherein the time scale of turbulence k/ϵ now replaces the meanflow time scale appearing in Eq. (2). The detailed discussion about the physical support for the inclusion of such a term appears in Ref. 5. The additional coefficient of the Richardson number term C_c, is taken as 0.2 on the basis of computer optimization here andin Ref. 5. It should be noted that, when the angular momentum of the meanflow increases with radius, Ri is positive, and therefore the new term acts to increase the level of ϵ and hence to reduce the turbulence kinetic energy. Consequently, the inclusion of the Ri-

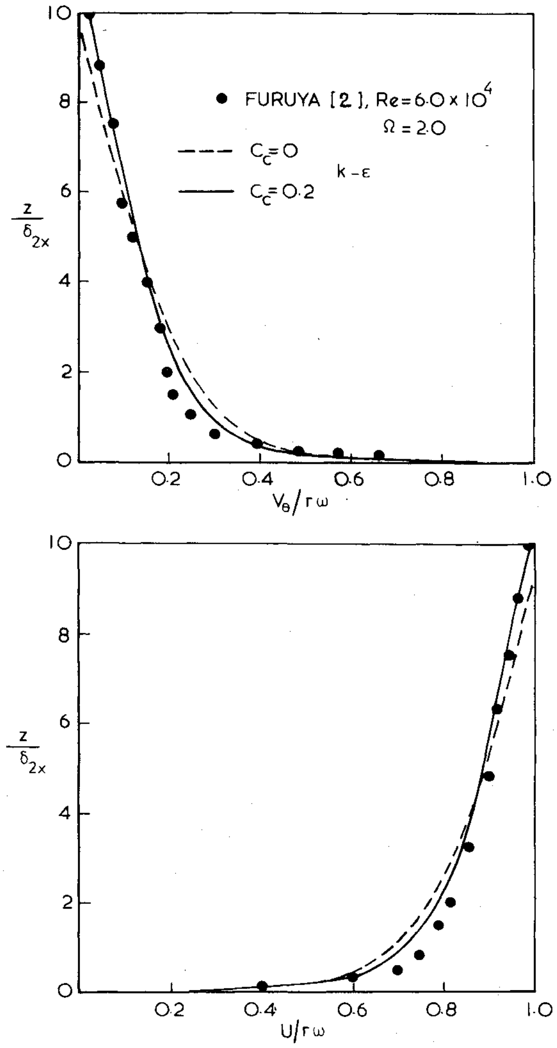

Fig. 1 Velocity profiles in flow past spinning cylinder.

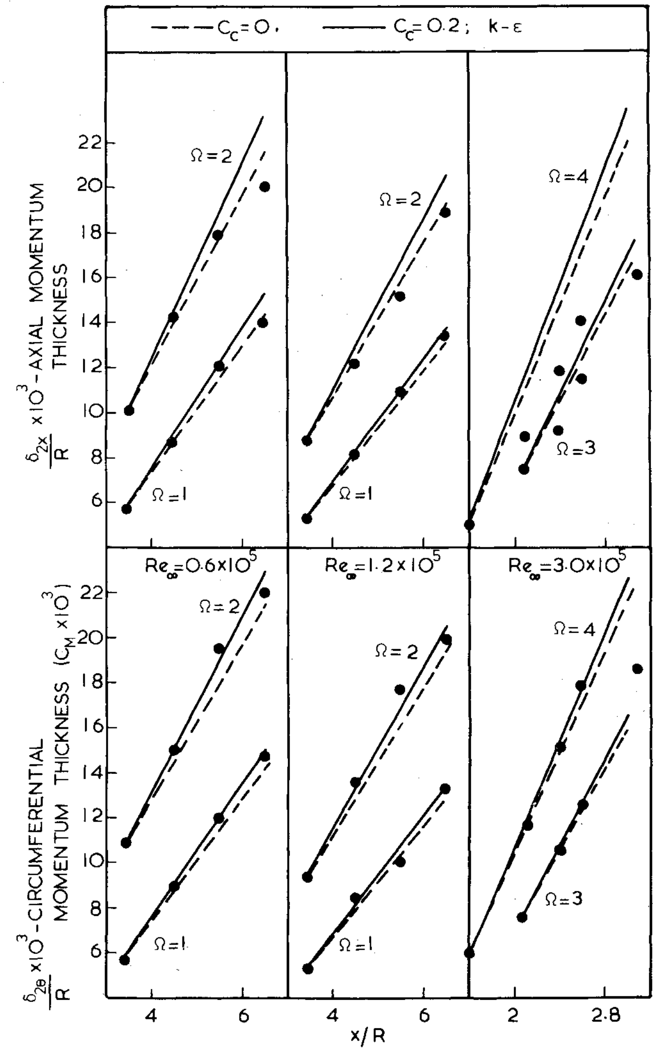

Fig. 2 Momentum thickness development for flow past spinning cylinder.

dependent term acts to diminish the turbulent viscosity. The reverse effects are produced in a flow where angular momentum is negative.

The above system of equations have been solved by the Patankar-Spalding[6] procedure modified for the inclusion of swirl as outlined in Ref. 1. Ninety grid nodes were used to span the boundary layer with a substantial concentration near the wall. The forward step used was typically 15% of the boundary-layer thickness leading to computer times per run of about 50 sec on a CDC 6600 computer.

Discussion of Predictions

Measured mean-velocity profiles for flow near a rotating cylinder in an axial stream by Furuya et al.[2] are shown in Fig. 1 for the case when the spin velocity at the cylinder surface is twice the freestream value. The ordinate z is normalized by the axial momentum thickness whereas surface velocity is used to normalize axial and tangential components of mean velocity. Predicted solutions with ($C_c = 0.2$) and without ($C_c = 0$) curvature correction term are shown. The predicted solutions with $C_c = 0.2$ are in closer agreement with the measured profiles over all of the fully turbulent region than the solutions with $C_c = 0$. The predicted level of axial velocity is still a little too low and that of swirl velocity somewhat too high for z/δ_{2x} between 0.5 and 2. The development of the axial and circumferential momentum thicknesses for three levels of Reynolds numbers is shown in Fig. 2. The rate of growth of circumferential momentum thickness at both levels of spin rates ($\Omega = 1$ and $\Omega = 2$) is predicted satisfactorily for the case where $C_c = 0.2$. However, the measured axial momentum thickness, although in satisfactory agreement at lower spin rates, grows less rapidly than the calculated values at higher spin rates, the discrepency being especially large for Parr's[3] data. To predict the correct rate of growth of both thicknesses, it would be necessary to abandon the use of the same coefficient of viscosity for axial and circumferential components.

Conclusions

The present inquiry has shown that the use of the energy-dissipation model of turbulence modified by the inclusion of the extra-source term to account for the effects of streamline curvature on the turbulent length scale leads to satisfactory agreement for flow past a rotating cylinder. The present model, however, predicts too fast a growth of axial momentum thickness at high levels of spin rates. To develop a model of turbulence for predicting swirling flows that possess significantly greater universality than the present one will require the abandonment of the isotropic viscosity concept. As the present level of disagreement is not large, it is questionable whether it is worthwhile to use what would be a much more elaborate turbulence closure for calculating this kind of flow.

Acknowledgment

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Boundary-Layer Separation on Moving Walls Using an Integral Theory

Kevin S. Fansler*
U.S. Army Ballistic Research Lab., Aberdeen Proving Ground, Md.

and

James E. Danberg†
University of Delaware, Newark, Del.

I. Introduction

THIS Note describes an integral method for obtaining separation from a moving wall. The technique originally was developed to obtain the force coefficients for a spinning cylinder in crossflow.[1] For handling the large boundary-layer reverse flows encountered on the spinning cylinder, the integral technique appears to have certain advantages over the finite-difference schemes.[2] With this method, the coupled integral-momentum and integral-energy equations are solved for the dependent variables, a shape factor K and the momentum thickness, in terms of the distance along the wall. The moving-wall similarity solutions provide a family of velocity profiles where all of the shape factors are known as functions of K and u_w/u_e (wall velocity to boundary-layer edge-velocity ratio); these shape factor relationships will be used to supply the values of the other shape factors appearing in the momentum and energy equations. These similarity solutions have been found for a wide range of u_w/u_e and pressure gradient parameter values.[3]

A new separation criterion is proposed here based on a singularity in the solution of the integral boundary-layer equations. The present criterion is a reasonable approximation to the Moore,[4] Rott,[5] and Sears[4] hypothesis for downstream moving walls and avoids some of the difficulties in using their criterion with the boundary-layer equations for upstream moving walls.[6]

Separation point predictions are compared with Vidal's[7] and Brady and Ludwig's[8] work. With the present approach, boundary layers with appreciable amounts of reverse flow can be calculated in a stable manner to separation.

II. Integral-Momentum and Integral-Energy Equations

The Prandtl equations and boundary conditions for steady, constant-property, incompressible, two-dimensional, boundary-layer flow over moving walls are assumed.[2] The radius a of the cylinder will be taken as a fundamental length for nondimensionalizing the physical coordinates. Here, x is the nondimensional coordinate along the surface, y is the result of multiplying the normal physical coordinate by $Re_d^{1/2}/a$ where $Re_d = 2u_0 a/v$. The kinematic viscosity is v. All velocities are nondimensionalized with respect to u_0, the freestream velocity, and with the y component of the physical velocity being stretched by the factor $Re_d^{1/2}$.

Integrating partially with respect to y, the integral momentum equation may be written as

$$\mathrm{d}\theta^2/\mathrm{d}x = 2[2T - (2+H)\theta^2(\mathrm{d}u_e/\mathrm{d}x)]/u_e \tag{1}$$

The nondimensionalized edge velocity is given by u_e and θ is the momentum thickness nondimensionalized in the same way as the normal physical coordinate. Here H is the ratio of the displacement thickness to the momentum thickness and the skin friction factor T is given by the expression $(\theta/u_e)(\partial u/\partial y)_w$. The corresponding integral energy equation is

$$\mathrm{d}(K^2\theta^2)/\mathrm{d}x = 2[4K(L + u_w T/u_e) - 3K^2\theta^2(\mathrm{d}u_e/\mathrm{d}x)]/u_e \tag{2}$$

Here K is the energy thickness divided by the momentum thickness and L is the dissipation integral nondimensionalized by multiplying with the factor $\tilde{\theta}/(\mu\tilde{u}_e^2)$ where the superscript tilde denotes dimensioned quantities. The two differential equations, (1) and (2), are coupled and can be solved simultaneously for the dependent variables, K and θ^2, as a function of x with suitable initial conditions.

III. Flow Separation from Moving Walls

Moore,[4] Rott,[5] and Sears[4] have suggested that $\partial u/\partial y = 0$, $u = 0$ as a criterion (*MRS* criterion for separation on moving walls). The experimental observations of Swanson[9] and of Brady and Ludwig[8] for separation on the downstream moving part of a rotating cylinder tend to support the *MRS* criterion. Numerical calculations by Telionis and Werle[10] of the boundary layer on a tilted parabola with a downstream moving wall also show a tendency toward an *MRS* profile as separation is approached. They found that the boundary-layer thickness grows rapidly in this region suggestive of the approach to a singularity similar to that found in the fixed-wall case. However, the *MRS* criterion cannot be applied directly for walls moving upstream because the boundary-layer equations cannot be balanced.[6] Furthermore, inspection of the reversed-flow similarity profiles reveals no special characteristics which can be associated with *MRS*-like separation, nor is there any tendency toward these conditions.[3]

Tsahalis[12] recently has studied a nonsimilar upstream-moving wall problem using a finite-difference method. Since the boundary-layer equations generally are not valid in the immediate neighborhood of separation, he argues that the full Navier-Stokes equations may permit an *MRS* profile. Thus, he concludes that an approach to "near *MRS*-like" profiles may be a signal of incipient separation although further confirmation of this hypothesis is needed.

In the present investigation, it is assumed that the self-similar velocity profiles provide an adequate variation in

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*Research Physicist, Exterior Ballistics Laboratory. Member AIAA.

†Professor of Mechanical and Aerospace Engineering. Associate Fellow AIAA.